

# Oscillating Universe in Hořava-Lifshitz Gravity

Kei-ichi MAEDA,<sup>1,2,\*</sup> Yosuke MISONOH,<sup>1,†</sup> and Tsutomu KOBAYASHI<sup>3,‡</sup>

<sup>1</sup>*Department of Physics, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan*

<sup>2</sup>*RISE, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan*

<sup>3</sup>*Research Center for the Early Universe (RESCEU), Graduate School of Science,  
The University of Tokyo, Tokyo 113-0033, Japan*

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We study the dynamics of isotropic and homogeneous universes in the generalized Hořava-Lifshitz gravity, and classify all possible evolutions of vacuum spacetime. In the case without the detailed balance condition, we find a variety of phase structures of vacuum spacetimes depending on the coupling constants as well as the spatial curvature  $K$  and a cosmological constant  $\Lambda$ . A bounce universe solution is obtained for  $\Lambda > 0, K = \pm 1$  or  $\Lambda = 0, K = -1$ , while an oscillation spacetime is found for  $\Lambda \geq 0, K = 1$ , or  $\Lambda < 0, K = \pm 1$ . We also propose a quantum tunneling scenario from an oscillating spacetime to an inflationary universe, resulting in a macroscopic cyclic universe.

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## I. INTRODUCTION

Recently Hořava proposed a power-counting renormalizable theory of gravity [1], which has attracted much attention over the past year. In Hořava's theory, Lorentz symmetry is broken and it exhibits a Lifshitz-like anisotropic scaling in the ultraviolet (UV),  $t \rightarrow \ell^z t, \vec{x} \rightarrow \ell \vec{x}$ , with the dynamical critical exponent  $z = 3$ . (For this reason the theory is called Hořava-Lifshitz (HL) gravity.) It is then natural to expect that the UV behavior of the theory would give rise to new scenarios of cosmology [2–4]. Earlier works have indeed revealed some interesting aspects of HL cosmology such as dark matter as integration constant [5], the generation of chiral gravitational waves from inflation [6], scale-invariant fluctuations without inflation [7], and possible dark energy scenario [8–10]. There are also some discussion closely related to observational cosmology and astrophysics such as cosmological perturbation [7, 11–22], observational constraints [23–25], primordial magnetic field without inflation [26], and a relativistic star [27, 28].

Though the viability of HL gravity is still under intense debate [5, 29–41], we give the theory the benefit of the doubt and will furthermore pursue consequences of Hořava's intriguing idea.

Here we focus on the dynamics of Friedmann-Lemaître-Robertson-Walker (FLRW) universe in HL gravity, which may provide us new aspect of the early universe. We study the FLRW spacetime with *non-zero* spatial curvature in the context of HL gravity. The non-trivial cosmological evolution is brought by the various terms in the potential which are constructed from the spatial curvature  $\mathcal{R}_{ij}$ . In particular, we find non-singular behavior in high curvature region, which may lead to avoidance of a

big bang initial singularity [4]. Consequently, many studies on the dynamics of the FLRW universe in HL gravity come out within the last few years [2–4, 42–60].

In the original HL gravity, the so-called detailed balance condition is assumed. However, this condition can be loosened to have arbitrary coupling constants [61]. Hence both models have been so far discussed in the analysis of the FLRW universe. As for matter components, a perfect fluid with the equation of state  $P = w\rho$  and a scalar field have been discussed. As a result, we can classify the analyzed models into four types: (1) The model with the detailed balance condition and with a perfect fluid [42–51], (2) The model with the detailed balance condition and with a scalar field [2, 4, 58] (3) The model without the detailed balance condition and with a perfect fluid [52–57, 60], and (4) The model without the detailed balance condition and with a scalar field [3, 59].

So far many works have been done assuming the detailed balance condition, and they reveal the possibility of singularity avoidance such as a bounce universe [2–4, 8, 21, 43, 44, 58] and an oscillating spacetime [43–45, 48]. The initial singularity is avoided because of “dark” radiation, which is a negative  $a^{-4}$  term. It comes from higher curvature terms. The “dark” radiation was first introduced in the context of a brane world [62]. Although such an effect is very interesting and important, the “dark” radiation term may fail to avoid a singularity if one include radiation or massless field. The conventional radiation behaves as  $a^{-4}$  with positive coefficient. If we have a sufficient amount of real radiation, the universe will inevitably collapse to a big-crunch singularity. Furthermore, if we assume that radiation field has also the same scaling law as gravity in the UV limit, the energy density of radiation field changes as  $a^{-6}$  [23], which is the same scaling law of stiff matter in the conventional theory. Inclusion of the positive  $a^{-6}$  term will kill the possibility of singularity avoidance by “dark” radiation. In order to save the present mechanism for singularity avoidance, one needs a negative  $a^{-6}$  term, which may be obtained in the generalized HL gravity model [61].

\*Electronic address: maeda@waseda.jp

†Electronic address: ym@underscore@misonou@moegi.waseda.jp

‡Electronic address: tsutomu@resceu.s.u-tokyo.ac.jp

Recently some papers have discussed the case without the detailed balance and studied a singularity avoidance (a bounce universe or an oscillating behavior): One is by use of a phase space analysis [53, 59], and the other is the case with perfect fluid with time-evolving equation of state [56, 60]. The former analysis was not properly performed because they introduce the dynamical variables more than the degrees of freedom. In the latter case, although they discuss some interesting transitions, the assumption of the equation of state is not so clear.

In the present paper, since there has so far not been a systematic and substantial analysis in cosmology based on this most general potential without the detailed balance condition, we provide a complete classification of the cosmological dynamics. We do not include any matter fields not only for simplicity but also to avoid unclear assumption. It is just straightforward to include perfect fluid with the equation of state  $P = w\rho$  ( $w=\text{constant}$ ). In particular, our analysis includes matter fluid with radiation and stiff matter as it is. We clarify which conditions should be satisfied for singularity avoidance. We also propose some possible scenario for a cyclic universe, i.e., the oscillating spacetime will transit by quantum tunneling to an inflationary phase, resulting in a cyclic universe after reheating.

The paper is organized as follows. After giving the generalized model of Hořava-Lifshitz gravity in §.II, we study the isotropic and homogeneous vacuum spacetime in §.III. We find a variety of phase structures including a bounce universe and an oscillating universe. We then invoke a more realistic cosmological model which may lead to a macroscopic cyclic universe via quantum tunneling from an oscillating universe.

## II. HOŘAVA-LIFSHITZ GRAVITY AND THE COUPLING CONSTANTS

The basic variables in HL gravity are the lapse function,  $N$ , the shift vector,  $N_i$ , and the spatial metric,  $g_{ij}$ . These variables are subject to the action [1, 61]

$$S = \frac{1}{2\kappa^2} \int dt d^3x \sqrt{g} N (\mathcal{L}_K - \mathcal{V}_{\text{HL}}[g_{ij}]), \quad (2.1)$$

where  $\kappa^2 = 1/M_{\text{PL}}^2$  and the kinetic term is given by

$$\mathcal{L}_K = \kappa_{ij} \kappa^{ij} - \lambda \kappa^2 \quad (2.2)$$

with

$$\kappa_{ij} := \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \quad (2.3)$$

being the extrinsic curvature. The potential term  $\mathcal{V}_{\text{HL}}$  will be defined shortly. In general relativity we have  $\lambda = 1$ , only for which the kinetic term is invariant under general coordinate transformations. In HL gravity, however, Lorentz symmetry is broken in exchange for renormalizability and the symmetry of the theory is invariance

under the foliation-preserving diffeomorphism transformations,

$$t \rightarrow \bar{t}(t), \quad x^i \rightarrow \bar{x}^i(t, x^j). \quad (2.4)$$

As implied by the symmetry (2.4) it is most natural to consider the projectable version of HL gravity, for which the lapse function is dependent only on  $t$ :  $N = N(t)$  [1]. Since the Hamiltonian constraint is derived from the variation with respect to the lapse function, in the projectable version of the theory the resultant constraint equation is not imposed locally at each point in space, but rather is an integration over the whole space. In the cosmological setting, the projectability condition results in an additional dust-like component in the Friedmann equation [see Eq. (3.2) below] [5].

The most general form of the potential  $\mathcal{V}_{\text{HL}}$  is given by [61]

$$\begin{aligned} \mathcal{V}_{\text{HL}} = & 2\Lambda + g_1 \mathcal{R} \\ & + \kappa^2 \left( g_2 \mathcal{R}^2 + g_3 \mathcal{R}^i_j \mathcal{R}^j_i \right) + \kappa^3 g_4 \epsilon^{ijk} \mathcal{R}_{il} \nabla_j \mathcal{R}^l_k \\ & + \kappa^4 \left( g_5 \mathcal{R}^3 + g_6 \mathcal{R}^i_j \mathcal{R}^j_i + g_7 \mathcal{R}^i_j \mathcal{R}^j_k \mathcal{R}^k_i \right. \\ & \left. + g_8 \mathcal{R} \Delta \mathcal{R} + g_9 \nabla_i \mathcal{R}_{jk} \nabla^i \mathcal{R}^{jk} \right), \end{aligned} \quad (2.5)$$

where  $\Lambda$  is a cosmological constant,  $\mathcal{R}^i_j$  and  $\mathcal{R}$  are the Ricci and scalar curvatures of the 3-metric  $g_{ij}$ , respectively, and  $g_i$ 's ( $i = 1, \dots, 9$ ) are the dimensionless coupling constants. (See Appendix A for some conditions on these coupling constants.)

In the original proposal [1] Hořava assumed the detailed balance condition, by which the potential term (2.5) is simplified to some extent. The potential under the detailed balance condition is given by

$$\begin{aligned} \mathcal{V}_{\text{DB}} = & -\frac{3\kappa^2 \mu^2 \Lambda_W^2}{2(3\lambda - 1)} + \frac{\kappa^2 \mu^2 \Lambda_W}{2(3\lambda - 1)} \mathcal{R} \\ & - \frac{(4\lambda - 1)\kappa^2 \mu^2}{8(3\lambda - 1)} \mathcal{R}^2 + \frac{\kappa^2 \mu^2}{2} \mathcal{R}^i_j \mathcal{R}^j_i \\ & - \frac{2\kappa^2 \mu}{\omega^2} c^{ij} \mathcal{R}_{ij} + \frac{2\kappa^2}{\omega^4} c_{ij} c^{ij}, \end{aligned} \quad (2.6)$$

where

$$c^{ij} := \epsilon^{ikl} \nabla_k \left( \mathcal{R}^j_l - \frac{1}{4} \mathcal{R} \delta^j_l \right) \quad (2.7)$$

is the Cotton tensor, and  $\Lambda_W$ ,  $\mu$  and  $\omega$  are constants. The potential (2.6) is therefore reproduced by identifying

$$\Lambda = -\frac{3(3\lambda - 1)}{2\mu^2 \kappa^2}, \quad (2.8)$$

$$g_1 = -1, \quad (2.9)$$

$$g_2 = -\frac{(4\lambda - 1)}{4(3\lambda - 1)} \mu^2 \kappa^2, \quad g_3 = \mu^2 \kappa^2, \quad (2.10)$$

$$g_4 = -\frac{4\mu \kappa^2}{\omega^2}, \quad g_5 = \frac{2\kappa^2}{\omega^4}, \quad g_6 = -\frac{10\kappa^2}{\omega^4},$$

$$g_7 = \frac{12\kappa^2}{\omega^4}, \quad g_8 = \frac{3\kappa^2}{2\omega^4}, \quad g_9 = \frac{4\kappa^2}{\omega^4}, \quad (2.11)$$

and  $\Lambda_W = -(3\lambda - 1)/(\mu^2 \kappa^2)$ . In the detailed balance case  $\mu$  and  $\omega$  are two free parameters.

In what follows, we adopt the unit of  $\kappa^2 = 1(M_{\text{PL}}^2 = 1)$  for brevity.

### III. FLRW UNIVERSE IN HOŘAVA-LIFSHITZ GRAVITY

We discuss an isotropic and homogeneous vacuum universe in Hořava-Lifshitz gravity. Note that such a vac-

uum spacetime is not realized in general relativity. We will extend our analysis to anisotropic spacetime (Bianchi cosmology) in the separate paper.

Assuming a FLRW spacetime, which metric is given by

$$ds^2 = -dt^2 + a^2 \left( \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right), \quad (3.1)$$

with  $K = 0$  or  $\pm 1$ . We find the Friedmann equation as

$$H^2 + \frac{2}{(3\lambda - 1)} \frac{K}{a^2} = \frac{2}{3(3\lambda - 1)} \left[ \Lambda + \frac{g_d}{a^3} + \frac{g_r}{a^4} + \frac{g_s}{a^6} \right], \quad (3.2)$$

where  $H = \dot{a}/a$ ,

$$\begin{aligned} g_d &:= 8C, \\ g_r &:= 6(g_3 + 3g_2)K^2, \\ g_s &:= 12(9g_5 + 3g_6 + g_7)K^3. \end{aligned} \quad (3.3)$$

A constant  $C$  may appear from the projectability condition and could be “dark matter”[5]. For a flat universe ( $K = 0$ ), the higher curvature terms do not give any contribution, and then the dynamics is almost trivial. Hence, in this paper, we discuss only non-flat universe ( $K = \pm 1$ ).

If  $\lambda = 1$ , we find a usual Friedmann equation for an isotropic and homogeneous universe in GR with a cosmological constant, dust, radiation and stiff matter. If  $g_d, g_r$ , and  $g_s$  are non-negative, such a spacetime gives a conventional FLRW universe model. However, since those coefficients come from higher curvature terms, their positivity is not guaranteed. Rather some of them could be negative. As a result, we find an unconventional cosmological scenario, which we shall discuss here. In what follows, we assume that  $\lambda > 1/3$ , but do not fix it to be unity.

In this paper, we assume  $C = 0$  just for simplicity. The Friedmann equation is written as

$$\frac{1}{2}\dot{a}^2 + u(a) = 0, \quad (3.4)$$

where

$$u(a) = \frac{1}{3\lambda - 1} \left[ K - \frac{\Lambda}{3}a^2 - \frac{g_r}{3a^2} - \frac{g_s}{3a^4} \right]. \quad (3.5)$$

Since the scale factor  $a$  changes as a particle with zero energy in this “potential”  $u$ , the condition  $u(a) \leq 0$  gives the possible range of  $a$  when the universe evolves. So we can classify the “motion” of the universe by the signs of  $K$  and  $\Lambda$ , and by the values of  $g_r$  and  $g_s$ . Note

that in the case with the detailed balance condition, we have

$$\begin{aligned} g_r &= 6(g_3 + 3g_2) = -\frac{3\mu^2}{2(3\lambda - 1)} < 0 \quad \text{for } \lambda > 1/3 \\ g_s &= 12(9g_5 + 3g_6 + g_7)K = 0. \end{aligned} \quad (3.6)$$

It is some special case of our analysis, although its dynamics will be completely different from generic cases because  $g_s$  vanishes.

We find mainly the following four types of the FLRW universe:

(1) [ $\mathcal{BB} \Rightarrow \mathcal{BC}$ ]: Suppose  $u(a) \leq 0$  for  $a \in (0, a_T]$ , and the equality is true only when  $a = a_T$ . A spacetime starts from a big bang ( $\mathcal{BB}$ ) and expands, but it eventually turns around at  $a = a_T$  to contract, finding a big crunch ( $\mathcal{BC}$ ).  $a_T$  is a scale factor when the universe turns around from expansion to contraction.

(2) [ $\mathcal{BB} \Rightarrow \infty$  or  $\infty \Rightarrow \mathcal{BC}$ ]: If  $u(a) < 0$  for any positive values of  $a$ , a spacetime starts from a big bang and expands forever, or its time reversal (A spacetime contracts to a big crunch). As for the asymptotic spacetime, we find  $a \propto t$  [Milne: $\mathcal{M}$ ] ( $K = -1$ ) for  $\Lambda = 0$ , while  $a \propto \exp(\sqrt{\Lambda/3}t)$  [de Sitter: $\mathcal{dS}$ ] for  $\Lambda > 0$ . We denote them as  $\mathcal{BB} \Rightarrow \mathcal{M}$ , and  $\mathcal{BB} \Rightarrow \mathcal{dS}$ , respectively. For the contracting cases, we describe them as  $\mathcal{M} \Rightarrow \mathcal{BC}$ , and  $\mathcal{dS} \Rightarrow \mathcal{BC}$ , respectively.

(3) [ $\mathcal{Bounce}$ ]: If  $u(a) \leq 0$  for  $a \in [a_T, \infty)$  and the equality holds only when  $a = a_T$ , a spacetime initially contracts from an infinite scale, and it eventually turns around at a finite scale  $a_T$ , and expands forever. The asymptotic spacetimes are the same as the case (2):  $\mathcal{M}$ , and  $\mathcal{dS}$ .

(4) [ $\mathcal{Oscillation}$ ]: If  $u(a) \leq 0$  for  $a \in [a_{\min}, a_{\max}]$  and the equality holds only when  $a = a_{\min}$  and  $a = a_{\max}$ , a spacetime oscillates between two finite scale factors.

For some specific values (or specific relations) of  $g_r$  and  $g_s$ , which divides two different phases of spacetimes, we

find a static universe ( $s$ ):

(5) [ $s$ ]: A spacetime is static with a constant scale factor  $a_S$ , if  $u(a_S) = 0$  and  $u'(a_S) = 0$ .

There are two types of static universes: one is stable ( $s_s$ ) and the other is unstable ( $s_u$ ). When we have an unstable static universe, we also find the following types of dynamical universes with a static spacetime as an asymptotic state as well:

(6) [ $s_u \Rightarrow \infty$  or  $\infty \Rightarrow s_u$ ]: If  $u(a) \leq 0$  for  $a \in [a_S, \infty)$  and the equality holds only at  $a_S$ , a spacetime starts from a static state in the infinite past, and expands forever, or it initially contracts from an infinite scale, and eventually reach a static state in the infinite future. We then have  $s_u \Rightarrow dS, \mathcal{M}$  or  $dS, \mathcal{M} \Rightarrow s_u$

(7) [ $\mathcal{B}\mathcal{B} \Rightarrow s_u$ , or  $s_u \Rightarrow \mathcal{B}\mathcal{C}$ ]: If  $u(a) \leq 0$  for  $a \in (0, a_S]$  and the equality holds only at  $a_S$ , A spacetime starts from a big bang and expands to a static state with a finite scale  $a_S$ , or its time reversal (A spacetime contracts from a static state to a big crunch).

(8) [ $s_u \Rightarrow \text{Bounce} \Rightarrow s_u$ ]: If  $u(a) \leq 0$  for  $a \in [a_S, a_T]$  (or  $a \in [a_T, a_S]$ ) and the equality holds only at  $a_S$  and  $a_T$ , a spacetime starts from a static state in the infinite past, and expands (or contracts). It eventually bounces at a finite scale  $a_T$ , and then reach a static state again in the infinite future.

For the case of  $\Lambda \neq 0$ , introducing the curvature scale  $\ell$  which is defined by

$$\frac{\Lambda}{3} = \frac{\epsilon}{\ell^2}, \quad (3.7)$$

where  $\epsilon = \pm 1$ , we can rescale the variables and rewrite the “potential”  $u$  by the rescaled variables as

$$u(a) = \frac{1}{3\lambda - 1} \left[ K - \epsilon \tilde{a}^2 - \frac{\tilde{g}_r}{3\tilde{a}^2} - \frac{\tilde{g}_s}{3\tilde{a}^4} \right], \quad (3.8)$$

where  $\tilde{a} = a/\ell$ ,  $\tilde{g}_r = g_r/\ell^2$ , and  $\tilde{g}_s = g_s/\ell^4$ . Using this potential and variables, we can discuss the fate of the universe without specifying the value of  $\Lambda$ .

A static universe will appear if we find a solution  $a = a_S (> 0)$  which satisfies  $u(a_S) = 0$  and  $u'(a_S) = 0$ . If  $\Lambda \neq 0$  ( $\epsilon = \pm 1$ ), it happens if there is a relation between  $\tilde{g}_r$  and  $\tilde{g}_s$ , which is defined by

$$\begin{aligned} \tilde{g}_s &= \tilde{g}_s^{[\epsilon, K](\pm)}(\tilde{g}_r) \\ &:= \frac{1}{9\epsilon^2} \left[ 2K - 3\epsilon K \tilde{g}_r \pm 2(1 - \epsilon \tilde{g}_r)^{3/2} \right]. \end{aligned} \quad (3.9)$$

This gives the curve  $\Gamma_{\epsilon, K(\pm)}$  on the  $\tilde{g}_r$ - $\tilde{g}_s$  plane, which gives the boundary between two different phases of spacetime. The radius of a static universe is given by

$$\tilde{a}_S = \tilde{a}_S^{[\epsilon, K](\pm)} := \sqrt{\frac{1}{3\epsilon} \left[ K \pm \sqrt{1 - \epsilon \tilde{g}_r} \right]}, \quad (3.10)$$

if it is real and positive. Here  $\pm$  correspond to the curves  $\Gamma_{\epsilon, (\pm)}$ . Since  $u(\tilde{a}) = 0$  is the cubic equation with respect

to  $\tilde{a}^2$  and  $\tilde{a}^2 = \tilde{a}_S^2$  is the double root, we have the third root, which is given by

$$\tilde{a}_T = \tilde{a}_T^{[\epsilon, K](\pm)} := \sqrt{\frac{1}{3\epsilon} \left[ K \mp 2\sqrt{1 - \epsilon \tilde{g}_r} \right]}, \quad (3.11)$$

where the universe turns around (or bounces). To exist such a point, it must be real and positive.

If  $\Lambda = 0$ , we find

$$g_s = -\frac{K}{12} g_r^2, \quad (3.12)$$

which is found from (3.9) in the limit of  $\epsilon = 0$ . The corresponding curve on the  $g_r$ - $g_s$  plane is denoted by  $\Gamma_{0, K}$ . The radius is given by

$$a_S = a_S^{[0, K]} := \sqrt{\frac{g_r}{6K}}, \quad (3.13)$$

assuming  $K g_r > 0$ .

Note that our classification depends just on  $g_r$  and  $g_s$  (or  $\tilde{g}_r$  and  $\tilde{g}_s$ ), apart from  $K$  and  $\Lambda$ . Since  $g_r$  and  $g_s$  are given by  $g_2, g_3, g_5, g_6$  and  $g_7$ , but do not include  $g_4, g_8$  and  $g_9$ , the fate of the universe is classified only by the conditions on the coupling constants of higher curvature terms but not on those of their derivatives such as  $\nabla_j \mathcal{R}_k^\ell$ .

In the case with the detailed balance conditions, we find  $\Lambda < 0$  from Eq. (2.8), and then obtain from Eq. (3.6)

$$\tilde{g}_r = -9/4, \quad \tilde{g}_s = 0. \quad (3.14)$$

Now we shall discuss what kind of spacetimes are realized under which conditions in the following three cases separately [**A.**  $\Lambda = 0$ , **B.**  $\Lambda > 0$ , and **C.**  $\Lambda < 0$ ].

$$\mathbf{A.} \quad \Lambda = 0$$

If a cosmological constant is absent, the “potential” is written as

$$u(a) = \frac{1}{(3\lambda - 1)a^4} \left[ K a^4 - \frac{g_r}{3} a^2 - \frac{g_s}{3} \right]. \quad (3.15)$$

In Fig. 1, we show the fate of the universe, which depends on the values of  $g_r$  and  $g_s$ . For the case of  $K = 1$ , there are two types of spacetime phases: One is  $\mathcal{B}\mathcal{B} \Rightarrow \mathcal{B}\mathcal{C}$ , and the other is an oscillating universe. In fact, if  $g_r > 0, g_s < 0$  and  $g_r^2 + 12g_s > 0$ , we find the scale factor  $a$  is bounded in a finite range as  $(0 <) a_{\min} \leq a \leq a_{\max} (< \infty)$ , where

$$\begin{aligned} a_{\min}^2 &\equiv \frac{1}{6} \left[ g_r - \sqrt{g_r^2 + 12g_s} \right] \\ a_{\max}^2 &\equiv \frac{1}{6} \left[ g_r + \sqrt{g_r^2 + 12g_s} \right], \end{aligned} \quad (3.16)$$

which gives an oscillating universe. The condition for an oscillating universe is written as

$$g_r > 0, \quad -\frac{g_r^2}{12} \leq g_s < 0, \quad (3.17)$$

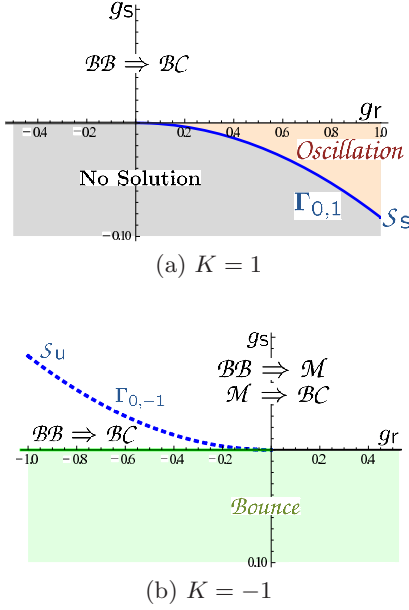


FIG. 1: Phase diagram of spacetimes for  $\Lambda = 0$ . The oscillating universe is found only for the case of  $K = 1$ . The stable and unstable static universes ( $s_s$  and  $s_u$ ) exist on the boundary  $\Gamma_{0,1}$  and  $\Gamma_{0,-1}$ , respectively. On  $\Gamma_{0,-1}$ , we also find dynamical universes with an asymptotically static spacetime;  $s_u \Rightarrow BC$ ,  $s_u \Rightarrow M$ ,  $BB \Rightarrow s_u$ , or  $M \Rightarrow s_u$ .

which is shown in Fig. 1(a) by “Oscillation” (the light-orange colored region) in the  $g_s$ - $g_r$  plane. The equality in Eq. (3.17), which is the curve  $\Gamma_{0,1}$ , gives a static universe with the scale factor  $a = a_S := \sqrt{g_r/6}$ .

For the case of  $K = -1$ , we find three types of space-time phases:  $BB \Rightarrow M$  (or  $M \Rightarrow BC$ ),  $BB \Rightarrow BC$ , and *Bounce* (see Fig.1(b)). On the boundary curve  $\Gamma_{0,-1}$ , which is defined by Eq. (3.12), i.e.,  $g_s = g_r^2/12$  ( $g_r < 0$ ), we find an unstable static universe  $s_u$ , and the dynamical universes with an asymptotically static spacetime;  $s_u \Rightarrow BC$ ,  $s_u \Rightarrow M$ ,  $BB \Rightarrow s_u$ , or  $M \Rightarrow s_u$ .

The bounce universe is found if  $g_s < 0$  or  $g_s = 0$  with  $g_r < 0$ , which is shown by “Bounce” (the light-green region). The radius at a turning point,  $a_T$  is given by

$$a_T = \sqrt{\frac{1}{6} \left( -g_r + \sqrt{g_r^2 - 12g_s} \right)} \quad (3.18)$$

Next we shall evaluate the period of an oscillating universe in the case of  $K = 1$ . The solution for Eq. (3.4) is given by

$$\begin{aligned} t - t_{\max} &= - \int_{a_{\max}}^a \frac{da}{\sqrt{-2u(a)}} \\ &= a_{\max} \sqrt{\frac{3\lambda - 1}{2}} E(\phi[a], k), \end{aligned} \quad (3.19)$$

where  $E(\phi, k)$  is the elliptic integral of the second kind,

which is defined by

$$E(\phi, k) := \int_0^\phi d\theta \sqrt{1 - k^2 \sin^2 \theta}. \quad (3.20)$$

$k$  and  $\phi[a]$  are given by

$$k := \frac{\sqrt{a_{\max}^2 - a_{\min}^2}}{a_{\max}}, \quad (3.21)$$

$$\phi[a] := \sin^{-1} \left( \frac{a_{\max}^2 - a^2}{a_{\max}^2 - a_{\min}^2} \right). \quad (3.22)$$

The period  $T$  is given by

$$T := 2(t_{\min} - t_{\max}) = 2a_{\max} \sqrt{\frac{3\lambda - 1}{2}} E(k), \quad (3.23)$$

where  $E(k)$  is the complete elliptic integral of the second kind defined by  $E(k) := E(\pi/2, k)$ .

In order to evaluate the period, we consider some limiting cases, which are the boundaries of the region of *Oscillation*. In Fig. 2, we show the potential  $u(a)$  by the blue curve for one boundary curve  $\Gamma_{0,1}$ , which is given by  $g_s = -g_r^2/12$ . It gives a stable static universe with the scale factor  $a_S$ . We also show the potential near the other boundary of *Oscillation* (the positive  $g_r$ -axis) by the dashed orange curve. Choosing, for example,  $g_r = 1$  and  $g_s = -0.001$ , we find an oscillating universe with the scale factor  $a \in [0.0316705, 0.576481]$ . Since these two poten-

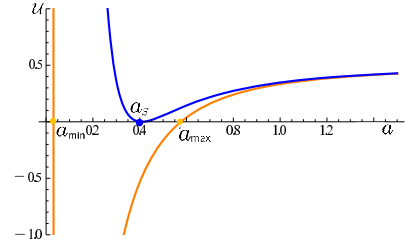


FIG. 2: The potential  $u(\tilde{a})$  for a stable static universe and an oscillating universe near the  $g_r$ -axis. The “coupling” constants are chosen as  $g_r = 1$  and  $g_s = -1/12$  on  $\Gamma_{0,1}$  for a static universe, which radius is shown by  $a_S = 1/\sqrt{6}$ . We also show the case with  $g_r = 1$  and  $g_s = -0.001$  for an oscillating universe, which maximum and minimum radii are given by  $a_{\max} = 0.576481$  and  $a_{\min} = 0.0316705$ , respectively.

tials give the limiting cases, we find that  $0 < a_{\min} \leq a_S$  and  $a_S \leq a_{\max} < \sqrt{2} a_S$  for an oscillating universe.

In the limit of a static universe (near  $\Gamma_{0,1}$ ), we find the period  $T_S$  as

$$T_S = \pi \sqrt{\left( \frac{3\lambda - 1}{2} \right) \frac{g_r}{6}}, \quad (3.24)$$

while in the other boundary limit ( $g_s \rightarrow 0$ ), we obtain

$$T_0 = \sqrt{\left( \frac{3\lambda - 1}{2} \right) \frac{4g_r}{3}}. \quad (3.25)$$



From these evaluations, giving the value of  $g_r$ , we find the period  $T$  of any oscillating universe is bounded in the range of  $(T_0, T_S)$  for  $g_s \in (-g_r^2/12, 0)$ . We then approximate the period as  $T \sim g_r^{1/2}$ .

We have found an oscillating FLRW universe because we have “negative” energy of “stiff matter” which comes from the higher curvature term. The condition for an oscillating universe is rewritten in terms of the original coupling constants as

$$g_3 + 3g_2 > 0, \quad (3.26)$$

$$-\frac{(g_3 + 3g_2)^2}{4} \leq 9g_5 + 3g_6 + g_7 < 0. \quad (3.27)$$

**B.  $\Lambda > 0$  ( $\epsilon = 1$ )**

In this case, the potential is given by

$$u(\tilde{a}) = \frac{1}{(3\lambda - 1)\tilde{a}^4} \left[ K\tilde{a}^4 - \tilde{a}^6 - \frac{\tilde{g}_r}{3}\tilde{a}^2 - \frac{\tilde{g}_s}{3} \right]. \quad (3.28)$$

For each value of  $K$ , we depict the fate of the universe in Fig 3, which depends on the values of  $\tilde{g}_r$  and  $\tilde{g}_s$ .

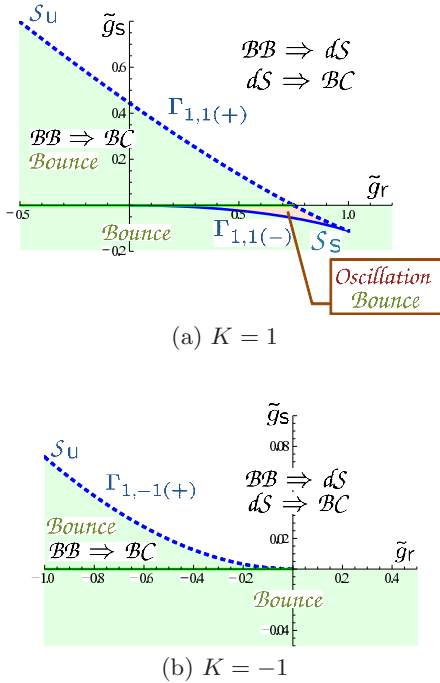


FIG. 3: Phase diagram of spacetimes for  $\Lambda > 0$ . The oscillating universe is found only for the case of  $K = 1$ . The static universes ( $s_s$  and  $s_u$ ) exist on the boundaries  $\Gamma_{1,K}(\pm)$ . We also find dynamical universes with an asymptotically static spacetime;  $s_u \Rightarrow dS$  or  $s_u \Rightarrow BC$  on  $\Gamma_{1,1}(+)$  ( $\tilde{g}_s \geq 0$ );  $s_u \Rightarrow dS$  or  $s_u \Rightarrow Bounce \Rightarrow s_u$  on  $\Gamma_{1,1}(+)$  ( $\tilde{g}_s < 0$ );  $s_u \Rightarrow dS$  or  $s_u \Rightarrow BC$  on  $\Gamma_{1,-1}$ .

We find non-singular evolution of the universe (*Bounce*, *Oscillation*, or *Static*) as well as the universe with a cosmological singularity ( $BB \Rightarrow BC$ ,  $BB \Rightarrow dS$ , or  $dS \Rightarrow BC$ ).

Except for the case of  $BB \Rightarrow BC$  and a static universe, the expanding universe approaches de Sitter spacetime (exponentially expanding universe) because of a positive cosmological constant  $\Lambda$ . The oscillating universe exists if and only if  $K = 1$  and the following conditions are satisfied:

$$\tilde{g}_r > 0 \quad (3.29)$$

$$\tilde{g}_s^{[1,1](-)}(\tilde{g}_r) \leq \tilde{g}_s \leq \tilde{g}_s^{[1,1](+)}(\tilde{g}_r), \quad (3.30)$$

where  $\tilde{g}_s^{[1,1](\pm)}$  is defined by Eq. (3.9) with  $\epsilon = 1, K = 1$ . This condition gives the constraint on  $\tilde{g}_s$  as  $-1/9 \leq \tilde{g}_s < 0$ . Note that in the limit of  $\tilde{g}_r \ll 1$  (i.e.  $\Lambda \rightarrow 0$ ), we recover the condition (3.17).

The boundaries of two different phases of spacetimes consist of the  $\tilde{g}_r$ -axis, and two curves ( $\Gamma_{1,1}(\pm)$ ) for  $K = 1$  or one curve ( $\Gamma_{1,-1}(+)$ ) for  $K = -1$ . Those boundary curves  $\Gamma_{1,K}(\pm)$  are defined by  $\tilde{g}_s = \tilde{g}_s^{[1,K](\pm)}(\tilde{g}_r)$ .

A stable static universe exists on the boundary curve  $\Gamma_{1,1}(-)$ , while unstable static universes appear on the boundary curves  $\Gamma_{1,\pm 1}(+)$ . For  $K = 1$ , there are two types of static universes (stable and unstable) corresponding to two curves  $\Gamma_{1,1}(-)$  and  $\Gamma_{1,1}(+)$ , respectively, which coincide at  $\tilde{g}_r = 1$  and  $\tilde{g}_s = -1/9$ . In the branches of unstable static universes ( $\Gamma_{1,K}(+)$ ), we also find dynamical universes with an asymptotically static spacetime;  $s_u \Rightarrow dS$  or  $s_u \Rightarrow BC$  on  $\Gamma_{1,1}(+)$  ( $\tilde{g}_s \geq 0$ );  $s_u \Rightarrow dS$  or  $s_u \Rightarrow Bounce \Rightarrow s_u$  on  $\Gamma_{1,1}(+)$  ( $\tilde{g}_s < 0$ );  $s_u \Rightarrow dS$  or  $s_u \Rightarrow BC$  on  $\Gamma_{1,-1}(+)$ .

The period  $T$  of an oscillating universe is calculated by

$$\tilde{T} := 2 \int_{\tilde{a}_{\min}}^{\tilde{a}_{\max}} \frac{d\tilde{a}}{\sqrt{-2u(\tilde{a})}}, \quad (3.31)$$

where  $\tilde{T} = T/\ell$ , and  $\tilde{a}_{\max}$  and  $\tilde{a}_{\min}$  are the maximum and minimum radii of the oscillating universe. We shall evaluate the period near the boundaries of the parameter range of oscillating universes (the light-orange region in Fig. 3(a)). We first show the potential  $u(\tilde{a})$  for three (near-) boundary values of  $\tilde{g}_s$  in Fig. 4

For the case with an unstable static universe (the dashed blue curve) ( $\Gamma_{1,1}(+)$  with  $\tilde{g}_s < 0$ ), the larger double root of the equation of  $u(\tilde{a}) = 0$  is given by

$$\tilde{a}_S = \tilde{a}_S^{[1,1](+)} := \sqrt{\frac{1}{3} \left( 1 + \sqrt{1 - \tilde{g}_r} \right)}, \quad (3.32)$$

while the smaller root is

$$\tilde{a}_T = \tilde{a}_T^{[1,1](+)} := \sqrt{\frac{1}{3} \left( 1 - 2\sqrt{1 - \tilde{g}_r} \right)}, \quad (3.33)$$

which corresponds to a turning radius at a bounce. The period  $T$  diverges in the limit of a static universe, because  $\tilde{a}_{\max} = \tilde{a}_S$  is the double root.

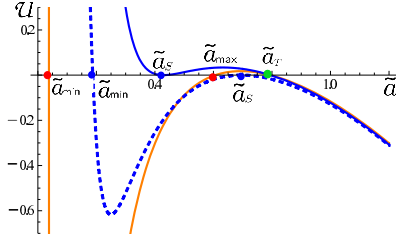


FIG. 4: The potential  $u(\tilde{a})$  for a stable and unstable static universes (the solid blue and the dashed blue), and that for an oscillating universe near  $\tilde{g}_r$ -axis (orange). The constants are chosen as  $\tilde{g}_r = 0.8$  and  $\tilde{g}_s = -0.0643206$  on  $\Gamma_{1,1(-)}$ , and  $-0.0245683$  on  $\Gamma_{1,1(+)}$ , for static universes, which radii are given by  $\tilde{a}_S$ , and  $\tilde{g}_r = 0.7$  and  $\tilde{g}_s = -0.001$  for an oscillating universe, which maximum and minimum radii are given by  $\tilde{a}_{\max}$  and  $\tilde{a}_{\min}$ , respectively. We also find  $S_u \Rightarrow \text{Bounce} \Rightarrow S_u$ , which bounce radius is given by  $\tilde{a}_{\min}$ .

While, near a stable static universe (the solid blue curve) ( $\Gamma_{1,1(-)}$ ), the period is finite and is evaluated as

$$\begin{aligned} \tilde{T}_S &= \left( \frac{3\lambda - 1}{2} \right)^{1/2} \times \pi \left[ \frac{1 - (1 - \tilde{g}_r)^{1/2}}{3(1 - \tilde{g}_r)^{1/2}} \right]^{1/2}, \\ &\approx \left( \frac{3\lambda - 1}{2} \right)^{1/2} \times \begin{cases} \frac{\pi}{\sqrt{6}} \tilde{g}_r^{1/2} & (\tilde{g}_r \ll 1) \\ \frac{\pi}{\sqrt{3}} \frac{1}{(1 - \tilde{g}_r)^{1/4}} & (\tilde{g}_r \approx 1). \end{cases} \end{aligned} \quad (3.34)$$

The period  $\tilde{T}_S$  changes from 0 to  $\infty$  along the static curve  $\Gamma_{1,1(-)}$ .

The radius of this stable static universe is given by  $\tilde{a}_S = \tilde{a}_S^{[1,1](-)}$ , which is the smaller root of the equation of  $u(\tilde{a}) = 0$ . The larger root  $\tilde{a}_T = \tilde{a}_S^{[1,1](+)}$  corresponds to a turning radius of a bounce universe, which is shown by  $\tilde{a}_T$  in Fig. 4.

There is another boundary limit, i.e.,  $\tilde{g}_s \rightarrow 0^-$ . In this limit, we find the roots of  $u(\tilde{a}) = 0$  as

$$\tilde{a}_1^2 \approx 0 \quad (3.35)$$

$$\tilde{a}_2^2 \approx \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4}{3}\tilde{g}_r} \right) \quad (3.36)$$

$$\tilde{a}_3^2 \approx \frac{1}{3} \left( 1 + \sqrt{1 - \frac{4}{3}\tilde{g}_r} \right). \quad (3.37)$$

Since the largest root ( $\tilde{a}_3$ ) corresponds to a turning radius  $\tilde{a}_T$  of a bounce universe, the oscillation range is  $[\tilde{a}_1, \tilde{a}_2]$ , and then the period is evaluated approximately by

$$\tilde{T}_0 = 2 \int_0^{\tilde{a}_2} \frac{d\tilde{a}}{\sqrt{-2u(\tilde{a})}}. \quad (3.38)$$

The period is then given by

$$\begin{aligned} \tilde{T}_0 &= \left( \frac{3\lambda - 1}{2} \right)^{1/2} \times 2 \sinh^{-1} \left[ \frac{1 - (1 - \frac{4}{3}\tilde{g}_r)^{1/2}}{2(1 - \frac{4}{3}\tilde{g}_r)^{1/2}} \right]^{1/2}, \\ &\approx \left( \frac{3\lambda - 1}{2} \right)^{1/2} \times \begin{cases} \frac{2}{\sqrt{3}} \tilde{g}_r^{1/2} & (\tilde{g}_r \ll 1) \\ \ln \left[ \frac{\sqrt{3}}{(\frac{3}{4} - \tilde{g}_r)^{1/2}} \right] & (\tilde{g}_r \approx \frac{3}{4}). \end{cases} \end{aligned} \quad (3.39)$$

The period  $\tilde{T}_0$  also changes from 0 to  $\infty$  along the  $\tilde{g}_r$ -axis;  $\tilde{g}_s = 0$  ( $0 < \tilde{g}_r < 3/4$ ).

We summarize our result as  $\tilde{T} \sim g_r^{1/2}$  when  $\tilde{g}_r \ll 1$ , but it diverges near  $\Gamma_{1,1(+)}$ , on which we have the unstable static universe.

### C. $\Lambda < 0$ ( $\epsilon = -1$ )

The potential is given by

$$u(a) = \frac{1}{(3\lambda - 1)\tilde{a}^4} \left[ K\tilde{a}^4 + \tilde{a}^6 - \frac{\tilde{g}_r}{3}\tilde{a}^2 - \frac{\tilde{g}_s}{3} \right], \quad (3.40)$$

We summarize our result in Fig. 5.

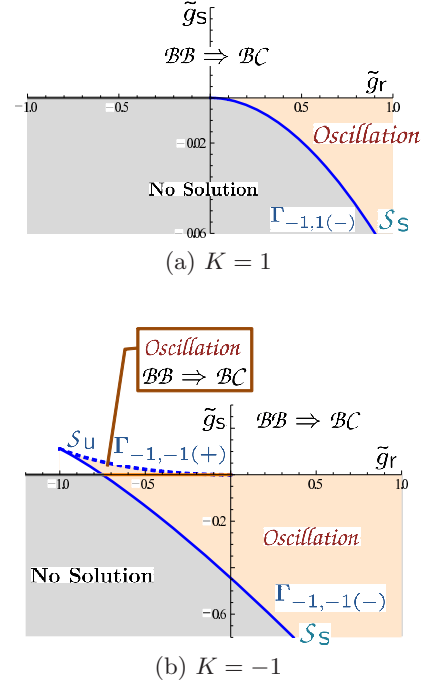


FIG. 5: Phase diagram of spacetimes for  $\Lambda < 0$ . The oscillating universe is found for both  $K = \pm 1$ . The static universe exists on the boundary  $\Gamma_{-1,1(-)}$  ( $K = 1$ ) and on  $\Gamma_{-1,-1(\pm)}$  ( $K = -1$ ). In the branch of unstable static universe on  $\Gamma_{-1,-1(+)}$ , we also find dynamical universes with an asymptotically static spacetime;  $\mathcal{B}\mathcal{B} \Rightarrow S_u$ ,  $S_u \Rightarrow \mathcal{B}\mathcal{C}$ , or  $S_u \Rightarrow \text{Bounce} \Rightarrow S_u$ .

In this case, if  $\tilde{g}_s > 0$ , we find a big bag and a big crunch singularities ( $\mathcal{BB} \Rightarrow \mathcal{BC}$ ) except for a small region in  $K = -1$ . If  $\tilde{g}_s < 0$ , however, we always find an oscillating universe if the solution exists.

The conditions for an oscillating universe is shown by the light-orange region in Fig. 5, which is given by the following inequalities:

For  $K = 1$ ,

$$\begin{aligned} \tilde{g}_r &> 0 \\ \tilde{g}_s^{[-1,1](-)}(\tilde{g}_r) &\leq \tilde{g}_s < 0, \end{aligned} \quad (3.41)$$

and for  $K = -1$ ,

$$\begin{aligned} \tilde{g}_s^{[-1,-1](-)}(\tilde{g}_r) &\leq \tilde{g}_s < 0 & \text{with } \tilde{g}_r \geq 0, \\ \tilde{g}_s^{[-1,-1](-)}(\tilde{g}_r) &\leq \tilde{g}_s \leq \tilde{g}_s^{[-1,-1](+)}(\tilde{g}_r) & \text{with } \tilde{g}_r < 0. \end{aligned} \quad (3.42)$$

In the limit of  $\tilde{g}_r \ll 1$  (i.e.  $\Lambda \rightarrow 0$ ) for  $K = 1$ , we recover the condition (3.17).

The boundary of the range of oscillating universe is given by the positive  $\tilde{g}_r$ -axis, and  $\Gamma_{-1,1(-)}$  for  $K = 1$ , and  $\Gamma_{-1,-1(\pm)}$  for  $K = -1$ . On those boundaries  $\Gamma_{-1,K(\pm)}$ , which are defined by  $\tilde{g}_s = \tilde{g}_s^{[-1,1](-)}(\tilde{g}_r)$  ( $K = 1$ ) and  $\tilde{g}_s = \tilde{g}_s^{[-1,-1](\pm)}(\tilde{g}_r)$  ( $K = -1$ ), we find a stable and unstable static universes.

The period of an oscillating universe is given by Eq.(3.31). We again evaluate its value near the boundary curves ( $\Gamma_{-1,K(-)}$ ) and the positive  $\tilde{g}_r$ -axis. The potentials  $u(\tilde{a})$  for the (near-) boundary values of  $\tilde{g}_s$  are shown in Fig. 6 ( $K = 1$ ), and Figs. 7 and 8 ( $K = -1$ ).

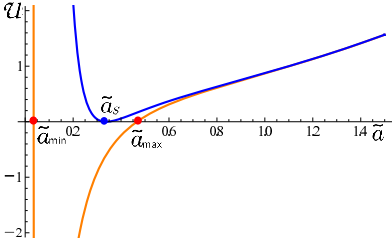


FIG. 6: The potential  $u(\tilde{a})$  for a stable static universe (blue) and an oscillating universe near  $\tilde{g}_r$ -axis (orange) in the case of  $K = 1$ . We set  $\tilde{g}_r = 0.8$  and  $\tilde{g}_s = -0.0477674$  on  $\Gamma_{-1,1}$  for a static universe with the radius  $\tilde{a}_S = 0.337461$ , and  $\tilde{g}_r = 0.8$  and  $\tilde{g}_s = -0.001$  for an oscillating universe, which maximum and minimum radii are given by  $\tilde{a}_{\max} = 0.466615$  and  $\tilde{a}_{\min} = 0.035439$ , respectively.

Note that the period diverges in the limit of an unstable static universe (on  $\Gamma_{-1,-1(+)}$ ), where we find the radius of a static universe by  $\tilde{a}_S = \tilde{a}_1$

$$\tilde{a}_1^2 := \frac{1}{3} \left( 1 - \sqrt{1 + \tilde{g}_r} \right). \quad (3.43)$$

The turning point is given by  $\tilde{a}_T = \tilde{a}_2$ , where

$$\tilde{a}_2^2 := \frac{1}{3} \left( 1 + 2\sqrt{1 + \tilde{g}_r} \right). \quad (3.44)$$

FIG. 7: The potential  $u(\tilde{a})$  for a stable static universe (blue) and an oscillating universe near  $\tilde{g}_r$ -axis (orange) for  $K = -1$ . We set  $\tilde{g}_r = 0.2$  and  $\tilde{g}_s = -0.581008$  on  $\Gamma_{-1,-1}$  for a static universe, which radius is given by  $\tilde{a}_S = 0.835752$ , and  $\tilde{g}_r = 0.2$  and  $\tilde{g}_s = -0.001$  for an oscillating universe, which maximum and minimum radii are given by  $\tilde{a}_{\max} = 1.03075$  and  $\tilde{a}_{\min} = 0.0683656$ , respectively.

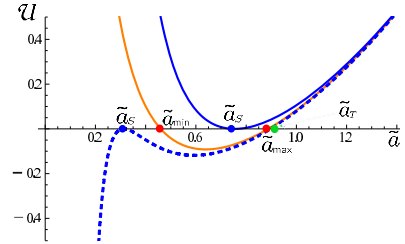


FIG. 8: The potential  $u(\tilde{a})$  for a stable and unstable static universes (blue and red, respectively) and an oscillating universe on  $\tilde{g}_r$ -axis (dashed orange) for  $K = -1$ . We set  $\tilde{g}_r = -0.5$  and  $\tilde{g}_s = -0.134123$  on  $\Gamma_{-1,-1(-)}$ , and  $\tilde{g}_s = 0.0230119$  on  $\Gamma_{-1,-1(+)}$  for static universes, which radius is given by  $\tilde{a}_S = 0.754344$ , and  $\tilde{g}_r = -0.5$  and  $\tilde{g}_s = 0$  for an oscillating universe, which maximum and minimum radii are given by  $\tilde{a}_{\max} = 0.888074$  and  $\tilde{a}_{\min} = 0.459701$ , respectively. We also find  $s_u \Rightarrow \text{Bounce} \Rightarrow s_u$ , which bounce radius is given by  $\tilde{a}_T = 0.897072$ .

Near a stable static universe ( $\Gamma_{-1,1(-)}$  and  $\Gamma_{-1,-1(-)}$ ), the period is evaluated as

$$\tilde{T}_S = \left( \frac{3\lambda - 1}{2} \right)^{1/2} \times \pi \left[ \frac{(1 + \tilde{g}_r)^{1/2} - K}{3(1 + \tilde{g}_r)^{1/2}} \right]^{1/2}, \quad (3.45)$$

which approaches a constant

$$\tilde{T}_S \approx \frac{\pi}{\sqrt{3}} \left( \frac{3\lambda - 1}{2} \right)^{1/2} \quad (3.46)$$

when  $\tilde{g}_r \gg 1$ .



Near the lower bound of  $\tilde{g}_r$ , we find

$$\tilde{T}_S \approx \frac{\pi}{\sqrt{3}} \left( \frac{3\lambda - 1}{2} \right)^{1/2} \times \begin{cases} \frac{\tilde{g}_r^{1/2}}{\sqrt{2}} & \rightarrow 0 \quad (\text{as } \tilde{g}_r \rightarrow 0 \text{ for } K = 1) \\ (1 + \tilde{g}_r)^{-1/4} & \rightarrow \infty \quad (\text{as } \tilde{g}_r \rightarrow -1 \text{ for } K = -1). \end{cases} \quad (3.47)$$

Hence the period  $T_S$  changes from 0 to a finite value (3.46) along the curve  $\Gamma_{-1,1(-)}$  for  $K = 1$ , while from  $\infty$  to at the same finite value along the curve  $\Gamma_{-1,-1(-)}$ .

The radius of a static universe is given by

$$\tilde{a}_S = \tilde{a}_S^{[-1,K](-)} = \sqrt{\frac{1}{3} (\sqrt{1 + \tilde{g}_r} - K)}. \quad (3.48)$$

In the case of  $\tilde{g}_r < -3/4$  with  $K = -1$ , there is another zero point of  $\mathcal{U}(\tilde{a})$ , which gives a maximum turning point of  $\mathcal{B}\mathcal{B} \Rightarrow \mathcal{B}\mathcal{C}$ , i.e.,

$$\tilde{a}_T = \tilde{a}_T^{[-1,-1](-)} = \sqrt{\frac{1}{3} (1 - 2\sqrt{1 + \tilde{g}_r})}. \quad (3.49)$$

Near  $\tilde{g}_r$ -axis, we find the solutions of the equation  $\mathcal{U}(\tilde{a}) = 0$  as

$$\tilde{a}_\pm^2 = \frac{1}{2} \left( -K \pm \sqrt{1 + \frac{4}{3}\tilde{g}_r} \right), \quad (3.50)$$

as well as  $\tilde{a}_0 \approx 0$ . We have a maximum radius  $\tilde{a}_{\max} = \tilde{a}_+$ , and find that the minimum radius  $\tilde{a}_{\min}$  is almost zero for  $\tilde{g}_r > 0$  because  $\tilde{a}_-^2 < 0$ , but in the case of  $K = -1$ , for  $-3/4 < \tilde{g}_r < 0$ , we find a finite minimum radius  $\tilde{a}_{\min} = \tilde{a}_-$ .

Using those values, we evaluate the period as

$$\tilde{T}_0 = \left( \frac{3\lambda - 1}{2} \right)^{\frac{1}{2}} \sec^{-1} \sqrt{1 + \frac{4}{3}\tilde{g}_r} \quad (3.51)$$

for  $K = 1$ , and

$$\tilde{T}_0 = \left( \frac{3\lambda - 1}{2} \right)^{\frac{1}{2}} \times \begin{cases} \pi - \sec^{-1} \sqrt{1 + \frac{4}{3}\tilde{g}_r} & (\tilde{g}_r \geq 0) \\ \pi & (-3/4 < \tilde{g}_r < 0) \end{cases}$$

for  $K = -1$ . The period  $\tilde{T}_0$  also changes from 0 to  $\infty$  along the  $\tilde{g}_r$ -axis;  $\tilde{g}_s = 0$  ( $0 < \tilde{g}_r < 3/4$ ).

In the case with the detailed balance condition, since  $\Lambda < 0$ ,  $\tilde{g}_r = -9/4$ ,  $\tilde{g}_s = 0$ , we do not find any FLRW solution. If we include matter fluid, the result will change. For example, if we have “radiation” fluid, which energy density is proportional to  $a^{-4}$ , we should shift the value of  $\tilde{g}_r$ . Then if  $-3/4 \leq \tilde{g}_r < 0$ , we find an oscillating universe for  $K = -1$ , which period is  $\pi[(3\lambda - 1)/2]^{1/2}$ . The equality ( $\tilde{g}_r = -3/4$ ) gives a static universe.

#### IV. TOWARD MORE REALISTIC COSMOLOGICAL MODEL

In the Hořava-Lifshitz gravity without the detailed balance condition, we find a variety of phase structures of vacuum spacetimes depending on the coupling constants  $g_r$  and  $g_s$  as well as the spatial curvature  $K$  and a cosmological constant  $\Lambda$ . Note that there is no vacuum FLRW solution in the case with the detailed balance condition. We summarize our result in Table I. We have obtained an oscillating spacetime as well as a bounce universe for a wide range of coupling constants. We have also evaluated the period of the oscillating universe.

	$K = 1$	$K = -1$
$\Lambda > 0$	$\begin{aligned} & * \text{Oscillation} \\ & * dS \longleftrightarrow \text{Bounce} \\ & * \mathcal{B}\mathcal{B} \Rightarrow \mathcal{B}\mathcal{C} \\ & * \mathcal{B}\mathcal{B} \Rightarrow dS \ (dS \Rightarrow \mathcal{B}\mathcal{C}) \end{aligned}$	$\begin{aligned} & * dS \longleftrightarrow \text{Bounce} \\ & * \mathcal{B}\mathcal{B} \Rightarrow \mathcal{B}\mathcal{C} \\ & * \mathcal{B}\mathcal{B} \Rightarrow dS \ (dS \Rightarrow \mathcal{B}\mathcal{C}) \end{aligned}$
	$\begin{aligned} & \Gamma_{1,1(\pm)} * S_u, S_s \\ & * \mathcal{B}\mathcal{B} \Rightarrow S_u \ (S_u \Rightarrow \mathcal{B}\mathcal{C}) \\ & * S_u \Rightarrow dS \ (dS \Rightarrow S_u) \\ & * S_u \longleftrightarrow \text{Bounce} \end{aligned}$	$\begin{aligned} & \Gamma_{1,-1(+)} * S_u \\ & * \mathcal{B}\mathcal{B} \Rightarrow S_u \ (S_u \Rightarrow \mathcal{B}\mathcal{C}) \\ & * S_u \Rightarrow dS \ (dS \Rightarrow S_u) \end{aligned}$
$\Lambda = 0$	$\begin{aligned} & * \text{Oscillation} \\ & * \mathcal{B}\mathcal{B} \Rightarrow \mathcal{B}\mathcal{C} \end{aligned}$	$\begin{aligned} & * \mathcal{M} \longleftrightarrow \text{Bounce} \\ & * \mathcal{B}\mathcal{B} \Rightarrow \mathcal{B}\mathcal{C} \\ & * \mathcal{B}\mathcal{B} \Rightarrow \mathcal{M} \ (\mathcal{M} \Rightarrow \mathcal{B}\mathcal{C}) \end{aligned}$
	$\begin{aligned} & \Gamma_{0,1} \\ & * S_s \end{aligned}$	$\begin{aligned} & \Gamma_{0,-1} * S_u \\ & * \mathcal{B}\mathcal{B} \Rightarrow S_u \ (S_u \Rightarrow \mathcal{B}\mathcal{C}) \\ & * S_u \Rightarrow \mathcal{M} \ (\mathcal{M} \Rightarrow S_u) \end{aligned}$
$\Lambda < 0$	$\begin{aligned} & * \text{Oscillation} \\ & * \mathcal{B}\mathcal{B} \Rightarrow \mathcal{B}\mathcal{C} \end{aligned}$	$\begin{aligned} & * \text{Oscillation} \\ & * \mathcal{B}\mathcal{B} \Rightarrow \mathcal{B}\mathcal{C} \end{aligned}$
	$\begin{aligned} & \Gamma_{-1,1(-)} \\ & * S_s \end{aligned}$	$\begin{aligned} & \Gamma_{-1,-1(\pm)} * S_u, S_s \\ & * \mathcal{B}\mathcal{B} \Rightarrow S_u \ (S_u \Rightarrow \mathcal{B}\mathcal{C}) \\ & * S_u \longleftrightarrow \text{Bounce} \end{aligned}$

TABLE I: Summary: What type of spacetime is possible for each  $\Lambda$  and each  $K$ . Non-singular universes are shown by the colored letters (an oscillating universe and dynamical spacetimes evolving in a finite scale range by red, static universes by blue, dynamical spacetimes evolving from or to an asymptotically infinite scale by green).  $dS$ ,  $\mathcal{B}\mathcal{B}$ ,  $\mathcal{B}\mathcal{C}$ ,  $S_u$ ,  $S_s$  and  $\mathcal{M}$  denote de Sitter space, a big bang, a big crunch, an unstable static universe, a stable static universe, and Milne universe, respectively.

In our analysis, we assume that the integration constant  $C$  from the projectability condition vanishes. If  $C \neq 0$ , one may find a different story. In fact, if  $g_s = 0$  and  $g_r < 0$  just as the case with the detailed balance condition, we will find the similar vacuum solutions to the present ones,

because  $C$  and  $g_r$  without  $g_s$ -term play the similar roles to those of  $g_r$  and  $g_s$  in the present model. For example, we obtain an oscillating universe for large  $C(> 0)$  with  $g_s = 0$ ,  $g_r < 0$ ,  $\Lambda = 0$  and  $K = 1$ . This avoidance of a singularity is, however, caused by the negative “radiation” density from the higher curvature terms. Hence if one includes the conventional radiation, then the effective  $g_r$  becomes positive as we will show below, and as a result the universe will inevitably collapse to a big-crunch singularity. Furthermore, if radiation field evolves as  $a^{-6}$  in the UV limit[23], the inclusion of such radiation will kill the possibility of singularity avoidance by “dark” radiation.

As we have evaluated, the oscillation period and amplitude are expected to be the Planck scale or the scale  $\ell$  defined by a cosmological constant  $\Lambda$ , unless the coupling constants are unnaturally large. Hence it cannot be a cyclic universe, which period is macroscopic such as the age of the universe.

In order to find more realistic universe, we have to include some other components, which we shall discuss here. First of all, one may claim inclusion of matter fluid. When we include a dust fluid ( $P = 0$ ), the conventional radiation ( $P = \rho/3$ ), and stiff matter ( $P = \rho$ ), we can treat such a case just by replacing the constant  $g_d$ ,  $g_r$  and  $g_s$  with

$$\begin{aligned} g_d &= 8C + g_{\text{dust}} \\ g_r &= 6(g_3 + 3g_2) + g_{\text{rad}} \\ g_s &= 12(9g_5 + 3g_6 + g_7)K + g_{\text{stiff}}, \end{aligned} \quad (4.1)$$

where  $g_{\text{dust}}$ ,  $g_{\text{rad}}$  and  $g_{\text{stiff}}$ , which come from real dust fluid, radiation and stiff matter, are positive constants. In this case, the present analysis is still valid. If  $g_{\text{rad}}$  is large enough just as our universe, a maximum scalar factor  $a_{\text{max}}$  of the the oscillating universe will become large (see, for example, Eq. (3.16)), and then it can be a cyclic universe.

If the equation of state is still given by  $P = w\rho$  ( $w=\text{constant}$ ), the analysis is straightforward. When we have other types of matter fields, e.g. a scalar field with a potential, the analysis will be more complicated. The phase space analysis may be appropriate for the case with a scalar field [63].

From our present analysis, one may speculate the following “realistic” scenario for the early stage of the universe. Suppose a closed universe is created from “nothing” initially in an oscillating phase (see Fig. 9) [64, 65]. Such a universe may be very small and oscillating between two radii ( $a_{\text{min}}$  and  $a_{\text{max}}$ ) with a time scale  $\ell$ . If we have a positive cosmological constant ( $\Lambda > 0$ ), there exists a potential barrier as shown in Fig. 9.

After numbers of oscillations, the universe may quantum mechanically tunnel to a bounce point  $a_T$ . Then the universe will expand to de Sitter phase because a positive cosmological constant, finding the universe in a

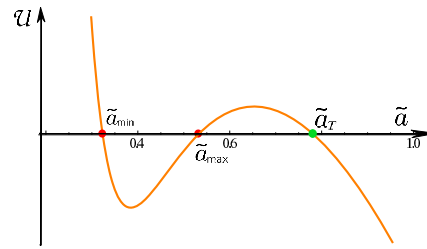


FIG. 9:

macroscopic scale<sup>1</sup>. Furthermore, one can refine this scenario, if there exists a scalar field, which is responsible for inflation, instead of a cosmological constant. Before tunneling, we may find the similar scenario to the above one. After tunneling, the potential of the scalar field will behaves as a cosmological constant in a slow-rolling period. We will find an exponential expansion of the universe after tunneling. However, inflation will eventually end and the energy of the scalar field is converted to that of conventional matter fluid via a reheating of the universe. We find a big bang universe. Since the universe is closed, but the scale factor has lower bound because of negative “stiff matter”, we will find a macroscopically large cyclic universe after all. To confirm such a scenario, we should analyze the dynamics of the universe with an inflaton field in detail. The work is in progress.

We also have another extension of the present FLRW spacetime to anisotropic one. It may be interesting and important not only to study the dynamics of Bianchi spacetime [66, 67] but also to analyze the stability of the FLRW universe against anisotropic perturbations[68].

## Acknowledgments

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## Appendix A: stability of a flat background and the coupling constants

In this Appendix, we discuss the conditions on the coupling constants by which gravitons are perturbatively sta-

<sup>1</sup> After we have written up this paper, we have found [60], in which a cosmological transition scenario from a static (or an oscillating) universe to an inflationary stage was discussed. They assume that the equation of state changes in time, which mechanism is not specified.

ble. From the perturbation analysis around a flat background, we obtain the dispersion relation for the usual helicity-2 polarizations of the graviton [17],

$$\omega_{\text{TT}(\pm)}^2 = -g_1 k^2 + g_3 \frac{k^4}{M_{\text{PL}}^2} \pm g_4 \frac{k^5}{M_{\text{PL}}^3} + g_9 \frac{k^6}{M_{\text{PL}}^4}. \quad (\text{A1})$$

The stability both in the IR and UV regimes requires

$$g_1 < 0, \quad g_9 > 0. \quad (\text{A2})$$

By a suitable rescaling of time, we then set  $g_1 = -1$ .

As a result of the reduced symmetry (2.4) the longitudinal degree of freedom of the graviton appears, and its stability is more subtle. First of all the longitudinal graviton is plagued with ghost instabilities for  $1/3 < \lambda < 1$  [1]. The dispersion relation for the longitudinal mode turns out to be [17]

$$\begin{aligned} \left( \frac{3\lambda - 1}{\lambda - 1} \right) \omega_L^2 = & g_1 k^2 + (8g_2 + 3g_3) \frac{k^4}{M_{\text{PL}}^2} \\ & + (-8g_8 + 3g_9) \frac{k^6}{M_{\text{PL}}^4}. \end{aligned} \quad (\text{A3})$$

We see that the sound speed squared is negative in the IR if  $g_1 < 0$  and  $\lambda > 1$ , which implies that the longitudinal graviton is unstable in the IR [36]. However, this fact itself does not necessarily mean that the theory suffers from pathologies, because whether or not an instability really causes a trouble depends upon its time scale [27]. Moreover, there is an attempt to improve the behavior of the longitudinal graviton by promoting  $N$  to an  $\vec{x}$ -dependent function and adding terms constructed from the 3-vector  $\partial_i N/N$  in the Lagrangian [35].<sup>2</sup> It can be shown that the non-projectable Hořava gravity thus extended appropriately does not plagued with instabilities of the longitudinal gravitons [35]. In light of these subtleties, we do not consider the stability of the longitudinal sector furthermore, while we do require the stability for the usual helicity-2 polarizations of the graviton.

Note that the detailed balance condition satisfies  $g_1 < 0$  and  $g_9 > 0$ .

## Appendix B: quantum tunneling from an oscillating universe

In the case of  $K = 1$  and  $\Lambda > 0$ , we have a bouncing universe as well as an oscillating universe. These two solutions are separated by a finite potential wall as we see in Fig 9. Hence we expect quantum tunneling from an oscillating universe to an exponentially expanding universe. In this Appendix, we shall evaluate the tunneling probability.

First we consider the normalized Euclidean metric

$$d\tilde{s}^2 = d\tilde{\tau}^2 + \tilde{b}^2(\tilde{\tau}) d\Sigma_{K=1}^2, \quad (\text{B1})$$

which satisfies the following equation

$$\tilde{b}'^2 - 2u(\tilde{b}) = 0, \quad (\text{B2})$$

where the prime denotes the derivative with respect to the Euclidean time  $\tilde{\tau}$ , and the potential  $u$  is written as

$$2u(\tilde{b}) = \frac{2}{3\lambda - 2} \frac{1}{\tilde{b}^4} \left[ -(\tilde{b}^2 - \tilde{b}_{\text{max}}^2)(\tilde{b}^2 - \tilde{b}_{\text{min}}^2)(\tilde{b}^2 - \tilde{b}_T^2) \right] \quad (\text{B3})$$

The variables with a tilde are normalized ones by use of the scale length  $\ell = \sqrt{3/\Lambda}$  just as in the text. The bounce solution  $\tilde{b}(\tilde{\tau})$  is obtained by integraton of Eq. (B2). The Euclidean action is given by

$$S_E = 3(3\lambda - 1)\ell \int d\tilde{\tau} d^3x \tilde{b} \left[ \frac{1}{2} \tilde{b}'^2 + u(\tilde{b}) \right]. \quad (\text{B4})$$

Using Eq. (B2), we find the action  $S_E$  as

$$S_E = 3(3\lambda - 1)\ell^2 V_3 \int d\tilde{b} \tilde{b} \sqrt{2u(\tilde{b})}, \quad (\text{B5})$$

where  $V_3 = 2\pi^2$  is the volume of a unit three sphere. Introducing  $u$  by

$$\tilde{b}^2 = \tilde{b}_T^2 (1 - k^2 u^2), \quad (\text{B6})$$

where  $k^2 = (\tilde{b}_T^2 - \tilde{b}_{\text{max}}^2)/\tilde{b}_T^2 (< 1)$ . We then find

$$\begin{aligned} S_E = & \frac{12\pi^2 \ell^2}{\kappa^2} (\tilde{b}_T^2 - \tilde{b}_{\text{max}}^2)^2 (\tilde{b}_T^2 - \tilde{b}_{\text{min}}^2)^{1/2} \\ & \times \int_0^1 \frac{u^2 du}{1 - k^2 u^2} \sqrt{(1 - u^2)(1 - m^2 u^2)}, \end{aligned} \quad (\text{B7})$$

where  $m^2 = (\tilde{b}_T^2 - \tilde{b}_{\text{max}}^2)/(\tilde{b}_T^2 - \tilde{b}_{\text{min}}^2) (< 1)$ .

It can be easily evaluated in the limit of a static universe, i.e.,  $\tilde{g}_s = \tilde{g}_s^{[1,1](-)}(\tilde{g}_r)$ . Using  $\tilde{b}_{\text{max}} \approx \tilde{b}_{\text{min}} \approx \tilde{b}_S$ , we find

$$\begin{aligned} S_E = & \frac{12\pi^2 \ell^2}{\kappa^2} (\tilde{b}_T^2 - \tilde{b}_S^2)^{5/2} \\ & \times \left( \frac{3 - 2k^2}{3k^4} - \frac{1 - k^2}{k^5} \tanh^{-1} k \right), \end{aligned} \quad (\text{B8})$$

where  $k = \sqrt{\tilde{b}_T^2 - \tilde{b}_S^2}/\tilde{b}_T$ . Since  $\tilde{b}_T^2 = (1 + 2\sqrt{1 - \tilde{g}_r})/3$  and  $\tilde{b}_T^2 - \tilde{b}_S^2 = \sqrt{1 - \tilde{g}_r}$ , we find

$$\begin{aligned} S_E = & \frac{4\pi^2 \ell^2}{\kappa^2} (1 - \tilde{g}_r)^{1/4} \\ & \times \left[ 1 - \frac{(1 + 2\sqrt{1 - \tilde{g}_r})^{1/2} (1 - \sqrt{1 - \tilde{g}_r})}{\sqrt{3}(1 - \tilde{g}_r)^{1/4}} \tanh^{-1} k \right], \end{aligned} \quad (\text{B9})$$

<sup>2</sup> Obviously, in this case the Hamiltonian constraint is imposed locally and the additional dust-like component does not appear in the Friedmann equation.

with

$$k^2 = \frac{3\sqrt{1-\tilde{g}_r}}{1+2\sqrt{1-\tilde{g}_r}}. \quad (\text{B10})$$

The tunneling probability is given by  $P \sim e^{-S_E}$ .

We show the behavior of  $S_E$  in Fig. 10. We find

$$\begin{aligned} P &\sim \exp \left[ -(20-40) \times \left( \frac{\ell}{\ell_{\text{PL}}} \right)^2 \right] \\ &\sim \exp \left[ -(60-120) \times \left( \frac{m_{\text{PL}}^4}{\rho_{\text{vac}}} \right) \right] \end{aligned} \quad (\text{B11})$$

except for two limiting cases:  $\tilde{g}_r \sim 1$ , in which  $S_E$  vanishes, and  $\tilde{g}_r \sim 0$ , in which  $S_E$  diverges. In the former case, the potential barrier vanishes giving a high tunneling probability, while in the latter case, the potential barrier diverges giving zero tunneling probability.

If the vacuum energy (or potential) just after tunneling is the Planck scale, the probability is evaluated as  $P \sim e^{-(60-120)}$ , which is very small but finite.

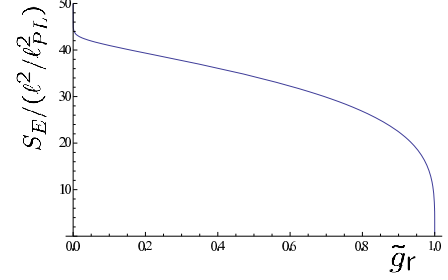


FIG. 10:

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